A Comparison of Control Techniques for Large Flexible Systems

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Two broad approaches to the active control of large flexible systems are compared, namely, coupled control and independent modal-space control. The basic difference between the two approaches lies in the manner in which the feedback controls are designed. The two methods are compared qualitatively from design and computational viewpoints, and quantitatively through the work done, performance index, and spillover effects. The comparison shows the independent modal-space control method to possess many advantages over coupled control, as it permits easier design and implementation and requires less computational effort and control energy for implementation.

Introduction

THE problem of control of flexible spacecraft has received a great deal of attention in recent years.[1,2] A satisfactory mathematical model for a flexible spacecraft may require a large number of degrees of freedom (>50), so that the question arises as to practical implementation of the control design. In particular, the question arises whether it is possible to control the spacecraft by means of on-board computers performing real-time computations.

A flexible spacecraft represents a distributed-parameter system of relatively complicated geometry. To control such a system, it is necessary to "discretize" it, i.e., to model it as a discrete system. It is customary to associate the order of the finite-dimensional discretized system with an equal number of modes and to refer to these modes as "modeled." Consistent with this, the balance of modes is referred to as "unmodeled." Any time a distributed system is discretized, errors occur. In fact, the eigensolution of the uncontrolled discretized system is only an approximation of the one of the distributed system, and the computed higher modes of the discretized system can be seriously in error.

There have been a number of control schemes proposed for flexible spacecraft control,[3-15] but they all represent one form or another of modal control. The term modal control must be interpreted in a very broad sense. Indeed, the implication is that a given number of spacecraft modes will be controlled. This amounts to designing the associated closed-loop eigenvalues so that the motion of the corresponding modes will be damped out. But, because design of feedback control is based on the open-loop eigensolution associated with the uncontrolled discretized system, the resulting closed-loop eigensolution is also an approximation, likely to have serious errors in the higher modes. The details of designing modal control vary from method to method, as does the implementation. Basic questions are the number of actuators and sensors required, the reliability of the computational algorithms for high-order systems, and the computer time necessary. The latter is very important, as it can make the difference between on-line and off-line computation.

Common methods of control are pole allocation,[16,17] linear optimal control,[18,19] and nonlinear on-off control.[20] In theory, at least, one can use a single actuator and a single sensor, provided the system is controllable and observable. But use of one actuator and one sensor is likely to induce so-called control and observation spillover, where the latter can cause instability.[21] Spillover refers to the phenomenon in which the energy intended to go into the controlled modes is pumped into the uncontrolled modes. One object of a good control design is to prevent spillover into the uncontrolled modeled modes, as there is no way of preventing spillover into the unmodeled modes. To minimize spillover into the unmodeled modes to the extent that the effect becomes insignificant, the order of the model must be quite large. This makes the implementation of the control methods mentioned above very difficult. For example, implementation of the pole allocation method is feasible mainly when only one actuator is used. Linear optimal control requires the solution of a matrix Riccati equation of an order equal to twice the number of controlled modes. Many different algorithms for the solution of the matrix Riccati equation have been proposed,[22-25] but all these algorithms experience computational difficulties for large-order systems. Nonlinear on-off controls can be implemented only for single input-single output cases.

A method capable of handling high-order systems which is extremely easy to apply is independent modal-space control. The method is based on the idea of coordinate transformations, whereby the system is decoupled into a set of independent second-order systems in terms of the modal coordinates. As a result, the control system design can be carried out for every second-order system independently, no matter how large the model is. The control gains thus obtained lead to modal control forces, i.e., some abstract forces corresponding to the modal coordinates. To obtain the actual forces, an inverse transformation is necessary. This procedure not only guarantees controllability but also guarantees that no control spillover into the modeled modes occurs, provided the number of actuators used is equal to the order of the discretized system. Observation spillover can be prevented by using a number of sensors capable of identifying all of the controlled modes.

In view of the above, it is convenient to refer to controls in which the system is not decoupled as "coupled control" and to the independent modal-space control as "independent control." Note that all control techniques fall into one of these two broad classes. It is clear that in coupled control one can use a relatively small number of actuators and sensors. However, the computational difficulties may manifest themselves not only in instability of the solution but also in excessive computational time, where the latter is likely to prevent on-line control implementation. Independent control has spillover and computational characteristics that are far superior to coupled control, but has high hardware requirements. More recently the robustness of the independent modal-space control method has been demonstrated analytically.[26] No such proof has yet been offered for coupled control.

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This paper is concerned with a comparison between coupled and independent control and attempts to provide answers to such questions as actuators locations and number, control energy, spillover characteristics, stability of control algorithms, computational time, etc. It is concluded that the independent modal-space control method is superior to coupled control from both design and computational viewpoints, as well as on the basis of quantitative considerations, such as input energy and on-line implementation.

Equations of Motion for Distributed-Parameter Systems

The equation of motion for a distributed-parameter system can be written as[27]

$$Lu(P,t)+M(P)\frac{\partial^2 u(P,t)}{\partial t^2}=f(P,t) \tag{1}$$

which must be satisfied at every point P of the domain D, where $u(P,t)$ is the displacement of point P, L a linear self-adjoint positive definite differential operator of order $2p$, $M(P)$ the distributed mass, and $f(P,t)$ the distributed controls. The displacement $u(P,t)$ is subject to the boundary conditions $B_i u(P,t)=0$ $(i=1,2,...,p)$, where B_i $(i=1,2,...,p)$ are linear differential operators. The associated eigenvalue problem consists of the differential equation

$$L\phi_r=\lambda_r M\phi_r, \qquad r=1,2,... \tag{2}$$

and the boundary conditions $B_i\phi_r=0$ $(i=1,2,...,p;$ $r=1,2,...)$. The eigensolution consists of a denumerably infinite set of eigenvalues λ_r $(r=1,2,...)$ and associated eigenfunctions ϕ_r. The eigenvalues are related to the natural frequencies ω_r of the undamped oscillation by $\lambda_r=\omega_r^2$ $(r=1,2,...)$. The eigenfunctions are orthogonal and can be normalized so as to satisfy $\int_D M\phi_r\phi_s \mathrm{d}D=\delta_{rs}$, $\int_D \phi_s L\phi_r \mathrm{d}D=\lambda_r\delta_{rs}$ $(r,s=1,2,...)$, where δ_{rs} is the Kronecker delta.

Using the expansion theorem

$$u(P,t)=\sum_{r=1}^{\infty}\phi_r(P)u_r(t) \tag{3}$$

where $u_r(t)$ are modal coordinates, Eq. (1) can be replaced by the infinite set of ordinary differential equations

$$\ddot{u}_r(t)+\omega_r^2 u_r(t)=f_r(t), \qquad r=1,2,... \tag{4a}$$

known as *modal equations,* where

$$f_r(t)=\int_D \phi_r(P)f(P,t)\mathrm{d}D, \qquad r=1,2,... \tag{4b}$$

are *modal control forces.*

Implicit in the preceding developments is the assumption that the full infinite set of eigenfunctions is readily available. This is true only in the case in which the eigenvalue problem admits a closed-form solution. In most practical cases, closed-form solutions are not possible, so one must be content with an approximate solution. This invariably requires that the numerical solution of an algebraic eigenvalue problem replace the differential eigenvalue problem [Eq. (2)]. But algebraic eigenvalue problems are defined over a finite-dimensional vector space, so that only a finite number of lower eigenvalues and eigenfunctions can be computed. Of the computed eigenvalues and eigenfunctions, at times less than one-half are accurate—and the accurate ones correspond to the lower eigenvalues.[27-29] Because it is seldom feasible to control the entire infinity of modes, we must truncate Eq. (3), which amounts to replacing a distributed-parameter model with a discrete one, a process known as discretization. Common sense dictates that only the accurate modes be included in the discretized model. Conversely, to obtain a discretized model of a given order, one must solve an eigenvalue problem of much higher order and truncate the model by retaining only the accurate modes. We shall refer to the finite number of modes included in the discretized model as *modeled modes,* which implies that there is an infinity of *unmodeled modes.*

Control System Design

Sometimes, the number of modes retained in the mathematical model may be very large. In such cases, it may not be feasible to control all of the modeled modes, so that the modeled modes are further divided into two categories, modeled controlled and modeled uncontrolled (residual). Hence, there are three sets of modes, namely, *modeled controlled, modeled uncontrolled,* and *unmodeled.*

It is assumed here that the modeled system is fully observable and that the modal coordinates corresponding to the controlled modes can be extracted from the system response with no observation spillover. The latter assumption can be justified when modal filters are used, both in the deterministic[13] and in the stochastic case.[30,31] Indeed, if a sufficient number of sensors is used, modal filters or modal filters in conjunction with independent modal-space Kalman filters can estimate the amplitudes belonging to any mode virtually free of observation spillover, both from modeled uncontrolled modes and from unmodeled modes. This makes it unnecessary to model uncontrolled modes so that we consider *only two classes of modes, modeled controlled and residual.* In this paper, we consider only the deterministic case, but the same conclusions are also valid in the stochastic case.

Equation (1) implies that the control force $f(P,t)$ is applied at every point P of the distributed-parameter system. Assuming that distributed control is not realizable, we propose to carry out the control task by means of discrete actuators.

Let us assume that there are m actuators. The actuator forces can be treated as distributed by writing

$$f(P,t)=\sum_{j=1}^{m}F_j(t)\delta(P-P_j) \tag{5}$$

where $F_j(t)$ are the actuator forces and $\delta(P-P_j)$ is a spatial Dirac delta function. Introducing Eq. (5) into Eq. (4b), we obtain

$$f_r(t)=\sum_{j=1}^{m}\phi_r(P_j)F_j(t), \qquad r=1,2,... \tag{6}$$

Equations (6) imply that the actual control forces $F_j(t)$ $(j=1,2,...,m)$ excite every single mode. Of course, these forces are selected so that the vibratory motion of the controlled modes is suppressed. In the process, the uncontrolled modes are also excited, a phenomenon that has come to be known as *control spillover.* If distributed controls were available, the modal control forces could be generated so as not to excite the residual modes. With discrete actuators, it must be assumed that spillover into the residual modes exists.[15]

Equations (4a) have the appearance of an infinite set of independent second-order differential equations and, in the absence of feedback control forces, they indeed are. This decoupling is referred to as *internal.* If the feedback forces $f_r(t)$ depend on all the modal coordinates

$$f_r=f_r(u_1,\dot{u}_1,u_2,\dot{u}_2,...), \qquad r=1,2,...,n \tag{7}$$

where n denotes the number of controlled modes, then Eqs. (4a) are *externally coupled.* This is the case of *coupled control.* In the special case in which f_r depends on u_r and $\dot{u}_r$ alone

$$f_r=f_r(u_r,\dot{u}_r), \qquad r=1,2,...,n \tag{8}$$

Equations (4a) become *internally and externally decoupled.* This is the essence of *independent modal-space control* (IMSC).[11-15]

The IMSC method permits the design of the control system for each mode separately, where the design takes place in the modal space. This is because when the control system design is based on the concept of IMSC, the modal control forces are computed first. The actual control forces are then obtained from the modal control forces by a linear transformation.

Next, let us introduce the vectors

$$u_C(t) = [u_1(t) \;\; u_2(t) \; ... \; u_n(t)]^T \tag{9a}$$

$$f(t) = [f_1(t) \;\; f_2(t) \; ... f_n(t)]^T \tag{9b}$$

$$F(t) = [F_1(t) \;\; F_2(t) \; ... \; F_m(t)]^T \tag{9c}$$

and the matrices

$$\Lambda_C = \text{diag}\,(\omega_1^2 \;\; \omega_2^2 \; ... \; \omega_n^2) \tag{10a}$$

$$B = [\phi_i(P_j)], i=1,2, ..., n; j=1,2,...,m \tag{10b}$$

Then, Eqs. (3) can be expressed as

$$\ddot{u}_C(t) + \Lambda_C u_C(t) = f(t) \tag{11}$$

where, from Eqs. (6),

$$f(t) = BF(t) \tag{12}$$

There are basic differences between coupled control and IMSC. In coupled control, one designs the control forces $F_j(t)$ $(j=1,2,...,m)$ in the actual space and the modal forces $f_r(t)$ $(r=1,2,...,n)$ do not appear explicitly in the formulation. The only restriction on the m number of actuators is that the discretized system be controllable, which requires that all of the components of the vector BF be different from zero, so that the amplitude of every controlled mode can indeed be driven to zero. Clearly, the modal equations are coupled by the feedback controls. On the other hand, in IMSC the modal forces $f_r(t)$ $(r=1,2,...,n)$ are designed first, which guarantees controllability by definition. Design of the modal controls f_r can be carried out independently of any other mode, which explains the name of the method. The actual control forces $F(t)$ can be synthesized from the modal control forces $f(t)$ by writing

$$F(t) = B^{\dagger} f(t) \tag{13}$$

where $B^{\dagger}$ is the pseudo-inverse of B. In general, the number of actuators is smaller than the number of controlled modes, $m<n$. But, pseudo-inverses are not actual inverses, so errors can be expected. This implies that the actual control vector $F(t)$ may not produce a modal control vector $f(t)$ as originally designed. In fact, at times the errors resulting from pseudo-inversion are sufficiently large to degrade the system performance. For this reason, we choose to write the inverse relation

$$F(t) = B^{-1} f(t) \tag{14}$$

which requires that *the number of actuators be equal to the number of controlled modes, $m=n$.* Because there is complete freedom in selecting the actuators locations, these locations should be chosen so that B is nonsingular.

Comparison of Coupled Control and Independent Modal-Space Control Methods

In coupled control, the number of actuators is not restricted, except for the requirement that the system be controllable. On the other hand, in IMSC the number of actuators must be equal to the number of controlled modes, but the feedback forces can be designed independently in the modal space. It turns out that this latter feature has important computational implications. In this section, we compare the design of some common feedback control techniques for coupled and independent control. To describe the various control techniques, it is convenient to recast the problem in state form. To this end, let us consider the auxiliary variable $v_r(t)$, defined by

$$v_r(t) = \dot{u}_r(t)/\omega_r, \qquad r=1,2,...,n \tag{15}$$

Then, introducing the state vector

$$w_C(t) = [u_1 \;\; v_1 \;\; u_2 \;\; v_2 \; ... \; u_n \;\; v_n]^T \tag{16}$$

and the matrices B' and A_C, where

$$B'_{2i-1,j} = 0, \quad B'_{2i,j} = \phi_i(P_j)/\omega_i, \quad i=1,2,...,n; \; j=1,2,...,m \tag{17a}$$

$$A_C = \text{block-diagonal}\,[A_1 \, A_2 \, ... \, A_n] \tag{17b}$$

in which

$$A_r = \begin{bmatrix} 0 & \omega_r \\ -\omega_r & 0 \end{bmatrix} \qquad r=1,2,...,n \tag{17c}$$

the modal equations can be expressed in the state form

$$\dot{w}_C(t) = A_C w_C(t) + B'F(t) \tag{18}$$

where we note that in general the term $B'F(t)$ provides external coupling.

Equation (18) can be rewritten as

$$\dot{w}_r(t) = A_r w_r(t) + B_r f_r(t), \qquad r=1,2,...,n \tag{19}$$

where

$$w_r(t) = [u_r(t) \;\; v_r(t)]^T, \qquad r=1,2,...,n \tag{20a}$$

$$B_r = [0 \; 1/\omega_r]^T, \qquad r=1,2,...,n \tag{20b}$$

in which $f_r(t)$ are defined by Eqs. (6). When IMSC is used, Eqs. (19) represent a set of independent equations, one for every controlled mode, so that internal and external decoupling is achieved. On the other hand, in the case of coupled control, Eqs. (19) are coupled by the feedback modal controls.

Methods of control in current use are: pole allocation, linear optimal control, and nonlinear on-off control. In the following, coupled control is compared with independent control for each of these techniques.

Pole Allocation

In this control technique, the closed-loop poles of the controlled modes are selected in advance and then control gains are computed so as to match the desired closed-loop poles.

For the case of coupled control, the actual control forces $F(t)$ can be expressed as

$$F(t) = Gw_C(t) \tag{21}$$

where G is a gain matrix of order $m \times 2n$. Introducing Eq. (21) into Eq. (18), we obtain the closed-loop state equation

$$\dot{w}_C(t) = (A_C + B'G)\, w_C(t) \tag{22}$$

It is shown in Ref. 32 that the elements of G can be obtained by solving the determinantal equations

$$\begin{vmatrix} \lambda_j - \rho_1 - \sum_{i=1}^{m} B'_{1i}G_{i1} & -\sum_{i=1}^{m} B'_{1i}G_{i2} & \cdots & -\sum_{i=1}^{m} B'_{1i}G_{i,2n} \\ -\sum_{i=1}^{m} B'_{2i}G_{i1} & \lambda_j - \rho_2 - \sum_{i=1}^{m} B'_{2i}G_{i2} & \cdots & -\sum_{i=1}^{m} B'_{2i}G_{i,2n} \\ \cdots & \cdots & \cdots & \cdots \\ -\sum_{i=1}^{m} B'_{2n,i}G_{i1}\ldots & -\sum_{i=1}^{m} B'_{2n,i}G_{i,2n} & \cdots & \lambda_j - \rho_{2n} - \sum_{i=1}^{m} B'_{2n,i}G_{i,2n} \end{vmatrix} = 0 \qquad j=1,2,\ldots,2n \tag{23}$$

where λ_j $(j=1,2,\ldots,2n)$ are the closed-loop poles and ρ_i $(i=1,2,\ldots,2n)$ are the open-loop poles. The expansion of this determinant for each desired closed-loop pole yields a set of $2n$ highly nonlinear algebraic equations for the gains G_{ij}. Furthermore, the system is undetermined, so that the determinantal equations must be augmented by additional equations for the gains, leading to construction of special types of controllers, such as "prescribed gain" and "minimum gain" controllers.[32] The difficulty involved in finding the gains G_{ij} by this approach is obvious. For large-order systems with multi-inputs, the method is not feasible. Only in the case of single-input systems can the control gains be determined in closed form.[32] A restricted approach to allocating the poles of a multi-input system is referred to as "dyadic control" and is originally due to Simon and Mitter.[16] The approach consists of designing a multi-input system as a single-input system by assuming the gain matrix G to be an outer (dyadic) product of two vectors, one of which is chosen arbitrarily, but still insuring the controllability of the modes, while the other vector (a row vector) is determined as the feedback gain of a single-input system by using the closed-form solution of Ref. 16. The resulting gain matrix G has always rank one and the control inputs are proportional to each other.

On the other hand, a pole allocation scheme for multi-input systems based on a transformation to a controllable canonical form is also essentially a single-input design, because the assignment of the poles is accomplished by only one of the inputs. The remaining inputs are used to set up a so-called cyclic form in which the system is transformed to a scalar (single-input) controllable form (see Ref. 17, Chap. 7). Therefore, pole allocation techniques tend to restrict control inputs, thus sacrificing the freedom to design a genuine multi-input control. The physical implications and consequences of such an approach for complex dynamical systems remains to be assessed, so that the use of the pole allocation technique for coupled control must be questioned. It can be concluded from the above that for the case of coupled control, the pole allocation technique is recommended only when a single actuator is used and a low number of modes is controlled.

By contrast, when independent control is used, the modal control forces yielding the desired closed-loop poles can be determined easily. Denoting these poles by

$$\lambda_r = \alpha_r \pm i\beta_r, \qquad r=1,2,\ldots,n \tag{24}$$

choosing the modal control forces $f_r(t)$ as

$$f_r(t) = a_r\omega_r u_r(t) + b_r\omega_r v_r(t), \qquad r=1,2,\ldots,n \tag{25}$$

where a_r and b_r $(r=1,2,\ldots,n)$ are the modal control gains and substituting Eq. (25) into Eq. (20b), we obtain the closed-loop equations

$$\dot{w}_r(t) = \begin{bmatrix} 0 & \omega_r \\ a_r - \omega_r & b_r \end{bmatrix} w_r(t), \qquad r=1,2,\ldots,n \tag{26}$$

Hence, the closed-loop poles are

$$\lambda_r = \frac{b_r}{2} \pm \frac{\sqrt{b_r^2 - 4\omega_r(\omega_r - a_r)}}{2} \tag{27}$$

Equating Eqs. (24) and (27), we obtain

$$a_r = (\omega_r^2 - \alpha_r^2 - \beta_r^2)/\omega_r, \quad b_r = 2\alpha_r, \quad r=1,2,\ldots,n \tag{28}$$

In view of the above, the IMSC method is found to be superior to coupled control when the pole allocation technique is used.

Linear Optimal Control

Optimal control implies that the feedback control forces are designed such that a certain performance index is minimized. We consider a performance index in the form

$$J = (w_C(t_f) - \hat{w})^T H (w_C(t_f) - \hat{w}) + \int_0^{t_f} [w_C^T(t) Q w_C(t) + R(t)]\,\mathrm{d}t \tag{29}$$

where t_f is the final time, $\hat{w}$ the desired state, and H and Q are weighting matrices. Moreover, $R(t)$ is a control cost term that can be expressed in terms of the modal control vector f_C and actual control vector F as

$$R(t) = F^T(t) R_1 F(t) \tag{30a}$$

$$R(t) = f^T(t) R_2 f(t) \tag{30b}$$

depending on the control method used. In the case of coupled control, one uses Eq. (30a), which is in terms of actual control forces. In the case of IMSC, one uses Eq. (30b), which is in terms of modal control forces. The weighting matrices R_1 and R_2 can be related to one another, as shown later. We consider the steady-state and transient cases of the regulator problem, in which case $\hat{w}=0$ and $H=0$. For the case of coupled control, minimization of J [Eq. (29) in conjunction with Eq. (30a)] leads to

$$F(t) = -R_1^{-1} B'^T K(t) w_C(t) \tag{31}$$

where the $2n \times 2n$ symmetric matrix K satisfies the matrix Riccati equation

$$\dot{K} = -KA_C - A_C^T K - Q - KB'R_1^{-1}B'^T K \tag{32}$$

The weighting matrix Q can be chosen as

$$Q = \text{diag}(\omega_1^2\ \omega_1^2\ \omega_2^2\ \omega_2^2 \ldots \omega_n^2\ \omega_n^2) \tag{33}$$

in which case $w_C^T Q w_C$ represents the Hamiltonian associated with the controlled modes. However, there are no guidelines for the choice of R_1. In addition, a certain choice of R_1 may not have a physical meaning, so that questions arise as to the nature of the performance index. The steady-state case, $\dot{K}=0$, implies that a set of nonlinear algebraic equations must be solved. Because K is symmetric, the number of nonlinear equations that must be solved is

$$N = 2n(2n+1)/2 = 2n^2 + n \tag{34}$$

As the number of controlled modes is increased, the number of equations to be solved simultaneously increases as the square of n. In the process, computational difficulties begin to surface.

Because the algebraic Riccati equation is nonlinear, its solution can be obtained iteratively.[22-25] Convergence and stability of an iterative algorithm cannot be guaranteed, and chances of obtaining convergent solutions decrease greatly as the order of the control system increases and as the number of actuators decreases. In fact, available algorithms can experience instabilities even for small-order systems.

One of the most modern and efficient algorithms for the solution of the algebraic Riccati equation is described in Ref. 25. Through a series of linear transformations, Ref. 25 reduces the iterative process to the eigensolution of a real general matrix. Even though stable algorithms that can handle general matrices have been developed in recent years,[33,34] it is well known that even these algorithms can experience difficulties in handling matrices of relatively high order.

It is estimated in Ref. 25 that for a Riccati equation of order p, $75p^3 + 0(p^2)$ multiplications are required for convergence. This number may increase for ill-conditioned matrices. For n controlled modes, the Riccati equation is of order $2n$, so that $[600n^3 + 0(n^2)]$ operations are required to determine the actual control forces $\boldsymbol{F}(t)$.

In the transient case, the $2n \times 2n$ nonlinear Riccati equation can be transformed into a linear set of equations of order $4n \times 4n$ to be integrated by an on-line process. The implementation of the transient solution of the Riccati equation requires ample computer capacity. Because this is an on-line process, such capacity may not be available.

When the IMSC method is used, J can be expressed as[12]

$$J = \sum_{r=1}^{n} J_r \tag{35}$$

where

$$J_r = \int_0^{t_f} (w_r^T Q_r w_r + \bar{R}_r f_r^2)\,dt, \qquad r=1,2,\ldots,n \tag{36}$$

is the performance index associated with the rth mode, Q_r a 2×2 weighting matrix, and $\bar{R}_r$ a control gain parameter. It turns out that J can be minimized by minimizing each and every J_r independently.[12] This decomposition is possible because each mode is controlled independently.

We realize from the above that R_2 in Eq. (30b) is a diagonal matrix of the form

$$R_2 = \text{diag}[\bar{R}_1\ \bar{R}_2 \ldots \bar{R}_n] \tag{37}$$

A logical choice for Q_r is $Q_r = \omega_r^2 I$, where I is the 2×2 identity matrix, so that $w_r^T Q_r w_r$ represents the Hamiltonian, i.e., the total energy associated with the rth mode.[12]

The minimization of J_r [Eq. (36)] leads to the modal feedback forces

$$f_r(t) = -\bar{R}_r^{-1} B_r^T K_r(t) w_r(t), \qquad r=1,2,\ldots,n \tag{38}$$

where the 2×2 symmetric matrix $K_r(t)$ satisfies the matrix Riccati equation[12]

$$\dot{K}_r = -K_r A_r - A_r^T K_r - Q_r + \bar{R}_r^{-1} K_r B_r B_r^T K_r \tag{39}$$

It is shown in Refs. 12-15 that *a closed-form solution of the Riccati equation exists for the steady-state case.* Therefore, the design of the feedback control gains does not represent a problem when the IMSC method is used. It is clear that the number of operations required for the solution of the Riccati equation is of order n. The total number of operations required to determine the actual control forces is $n^3/2 + 0(n^2)$. The $n^3/2$ operations are required to obtain the inverse of the $n\times n$ matrix B (see Ref. 29). As a result, even with the most advanced algorithms for the solution of the Riccati equation, coupled control requires 1200 times more operations than independent control to determine optimal control gains. In addition, for cases when the closed-loop poles are changed without changing the order of the control system and actuators locations, independent control requires virtually no computational effort, because the B matrix remains unchanged.

For the transient case, the 2×2 nonlinear Riccati equation can be replaced by a linear equation of order 4×4. As a result, one must integrate n sets of equations of order 4×4 by an on-line process. This task is much simpler and requires much less computer capacity than integrating a $4n\times 4n$ matrix on-line.

Nonlinear On-Off Control

On-off controls provide for a region of deadband, based on the recognition that within some small bounds oscillations are tolerable.[20] Because of the deadband region, the actuators need not operate continuously.

For the most part, on-off control has been implemented for single input-single output systems. The implementation makes use of switching curves in the phase plane. For multi input-multi output systems, one would have to use switching surfaces in the phase space, which is not feasible.[18] Hence, on-off control in coupled form is virtually impossible.

By contrast, when the IMSC method is used, the design of on-off controls can be accomplished with relative ease. The modal force for the rth mode is given by[20]

$$f_r = \begin{cases} -k_r, & v_r > d_r \\ 0, & -d_r \le v_r \le d_r, \\ k_r, & v_r < -d_r \end{cases} \qquad r=1,2,\ldots,n \tag{40}$$

where k_r is a control gain parameter and $2d_r$ the magnitude of the deadband region. Their values can be determined readily for each mode. Reference 20 also gives closed-form expressions for the response of the modal coordinates.

It is clear that if nonlinear on-off controls are desired, IMSC is the only method available. Note that the on-off controls are modal controls. The actual controls are quantized, i.e., they represent linear combinations of on-off functions.

Control Implementation

It is clear from the previous section that independent control is quite often more desirable than coupled control from a design viewpoint. The computational problems associated with coupled control increase appreciably as the order of the control system is increased. Coupled control requires a great deal of computer time and storage. In fact, for large-order systems, storage capabilities may be excessive, so that a large computer may be required. On the other hand, independent control requires much less computational effort and computer storage, so that it can be implemented by microcomputers.

The work done to control a distributed-parameter system has the expression

$$W=\int_D \left[\int_{u_0}^{u_f} f(P,t)\,\mathrm{d}u(P,t)\right]\mathrm{d}D \tag{41}$$

where u_0 and u_f are the initial and final displacements, respectively. Substituting Eqs. (3) and (5) into Eq. (41) we obtain[35]

$$W=\sum_{r=1}^{\infty}\int_{u_{r0}}^{u_{rf}}\sum_{j=1}^{m}F_j(t)\phi_r(P_j)\,\mathrm{d}u_r(t) \tag{42}$$

The work done to control the controlled modes, W_C, is given by

$$W_C=\sum_{r=1}^{n}\int_{u_{r0}}^{u_{rf}}\sum_{j=1}^{m}F_j(t)\phi_r(P_j)\,\mathrm{d}u_r(t) \tag{43}$$

or, considering Eqs. (9a), (9c) and (10b),

$$W_C=\int_{u_{C0}}^{u_{Cf}} F^T B^T \mathrm{d}u_C \tag{44}$$

Equation (44) is valid for all control methods. In the case of IMSC, substituting Eq. (14) into Eq. (44), we obtain

$$W_C=\int_{u_{C0}}^{u_{Cf}} f^T(t)\,\mathrm{d}u_C(t)=\int_0^{t_f} f^T(t)\,\dot{u}_C(t)\,\mathrm{d}t \tag{45}$$

Because in the IMSC method the modal control forces $f(t)$ are determined independently of the actuators locations, Eq. (45) indicates that *in the case of IMSC the work done to control the controlled modes does not depend on the actuator locations.*

Because in coupled control one determines the control vector F and not the modal control vector f, we conclude from Eq. (44) that in the case of coupled control the work done to control the controlled modes does depend on the actuator locations and their number. Hence, to find a global minimum, one must adopt a search process for the optimal location of the actuators and their number. The control gains also depend on the actuator locations, so that any minimum obtained on the basis of a given number of actuators is only a local minimum, i.e., a constrained minimum for the particular number or particular locations of actuators. The global minimum for the energy imparted to the distributed system is obtained in the case of the independent modal-space control method, which is subject to no constraints. The IMSC method leads to a global minimum for the energy, and the actuators locations are immaterial as far as the controlled modes are concerned. One can design the actuators locations such that the energy pumped into the residual modes is minimized.[35]

A similar argument concerning global and local minima can be made for the case of optimal control. When the IMSC method is used, the performance index J is independent of the actuators locations. However, when coupled control is used, J does depend on the number and locations of the actuators, so that minimization of J is a constrained one. Hence, only a local minimum can be achieved for coupled controls. On the other hand, for independent control the minimization of J is unconstrained, so that a global minimum can be obtained.

In view of the above, the advantage of coupled control over IMSC lies in the smaller number of actuators, provided controllability is insured. Note that controllability is guaranteed by definition for IMSC. On the other hand, the advantages of IMSC over coupled control are many and can be listed as follows: 1) larger choice of control techniques, including nonlinear control, 2) less computational effort, 3) less computer storage requirement, 4) capability of implementation by microcomputers, 5) easier design, 6) virtually no instability characteristics, 7) less energy used, and 8) actuators locations do not affect the work performed to control the controlled modes. Some of these advantages are illustrated via a numerical example in the following section.

Numerical Example

To compare the two control concepts quantitatively, we consider control of the axial vibration of a tapered bar of length 10 fixed at one end and free at the other. We choose the mass and stiffness operators

$$M(x)=2(1-x/10) \tag{46a}$$

$$L=-\frac{\mathrm{d}}{\mathrm{d}x}\left[EA(x)\frac{\mathrm{d}}{\mathrm{d}x}\right]=-2\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(1-\frac{x}{10}\right)\frac{\mathrm{d}}{\mathrm{d}x}\right] \tag{46b}$$

The boundary conditions are

$$B_1(0)=1 \tag{47a}$$

$$B_1(10)=EA(10)\frac{\mathrm{d}}{\mathrm{d}x} \tag{47b}$$

It can be shown that the associated eigenvalue problem lends itself to a closed-form solution. The transcendental equation takes the form

$$J_0(10\omega)=0 \tag{48}$$

where J_0 is the Bessel function of zero order and of the first kind and ω the natural frequency. The solution of Eq. (48) can be found in many mathematical tables (see, for example, Ref. 36, p. 469, Table 9.5). Table 1 lists the first 12 natural frequencies associated with the tapered bar in question.

It can also be shown that the normalized eigenfunctions have the form

$$\phi_r(x)=\frac{J_0[\omega_r(10-x)]}{\sqrt{10}J_1(10\omega_r)},\qquad r=1,2,\ldots \tag{49}$$

where J_1 is the Bessel function of the first order and of the first kind.

During all simulations, the system response was obtained by a transition matrix approach, using a sampling period of 0.02 s. The response was obtained for an excitation in the form of a unit impulse applied at an arbitrary point x_0, with the initial displacement and velocity taken as zero. The unit impulse excitation can be expressed as

$$f_e(x,t)=F_0\delta(x-x_0)\delta(t) \tag{50}$$

It can easily be verified that the unit impulse results in the initial modal velocities

$$\dot{u}_r(0)=F_0\phi_r(x_0),\qquad r=1,2,\ldots \tag{51}$$

Table 1 Natural frequencies

ω_1	0.2405	ω_5	1.4931	ω_9	2.7493
ω_2	0.5520	ω_6	1.8071	ω_{10}	3.0635
ω_3	0.8654	ω_7	2.1212	ω_{11}	3.3778
ω_4	1.1792	ω_8	2.4352	ω_{12}	3.6917

The point at which the impulse was applied was chosen as $x_0 = 6.3$. It was assumed that a sufficient number of sensors was used, so that the modal filters extracted the amplitudes of the controlled modes accurately.

For all simulations, the actuators locations were taken as

$$x_i^a = 10i/(m+1), \qquad i = 1,2,\ldots,m \tag{52}$$

In the following, we implement the three control techniques described earlier by using coupled control and independent control and compare the system performance.

Pole Allocation Technique

As pointed out earlier, in the case of coupled control the pole allocation technique is feasible only when a single actuator is used and a very small number of modes is modeled. Hence, we make no attempt to use more than one actuator. By contrast, when independent control is used, so that the number of actuators is equal to the number of controlled modes, the implementation of the pole allocation technique is relatively simple.

Let us control five modes by the pole allocation method. The desired closed-loop poles are $-0.09 \pm i0.24$, $-0.10 \pm i\,0.55$, $-0.09 \pm i0.87$, $-0.11 \pm i1.18$, and $-0.11 \pm i1.48$. The actual control forces were obtained for coupled control for the case when a single actuator was used.

Let us first compare the response of the bar. Figure 1 compares the contribution to the displacement of the tip of the bar, $u(10,t)$, of the five controlled modes for the cases of coupled and independent controls. Even though the closed-loop poles are the same for both cases, it is seen that using five actuators regulates the vibration of the tip of the bar faster than using one actuator. This is because when one actuator is used, the modal feedback control forces recouple the controlled modes, so that there is energy transfer from one mode to another, thus degrading the system performance.

Next, we compare the amount of work done to control the controlled modes. The energy imparted to the distributed system can be calculated by using Eq. (42). Figure 2 compares the work done to control the controlled modes for the cases of one and five actuators. As can be easily observed, the use of independent control is more energy efficient than the use of coupled control, at least for the example considered. In addition, when using a single actuator, one is limited by the order of the controller for which gains can be computed.

Linear Optimal Control

In linear optimal control, the feedback control forces are selected so that a quadratic cost function is minimized.[18] A comparison of independent control with coupled control must be based on comparable weighting matrices for the performance measure. Whereas the weighting matrix Q for the state vector can be the same for both cases, the weighting term $R(t)$ for the control vector must be different. For independent control $R(t)$ is chosen in terms of the modal

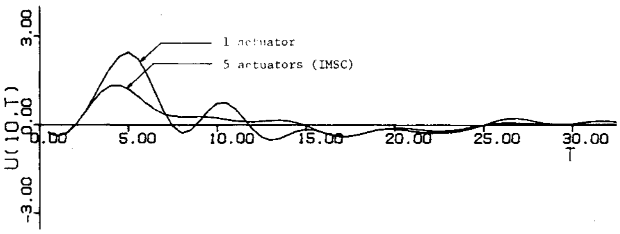

Fig. 1 Displacement at the end of the bar, pole allocation.

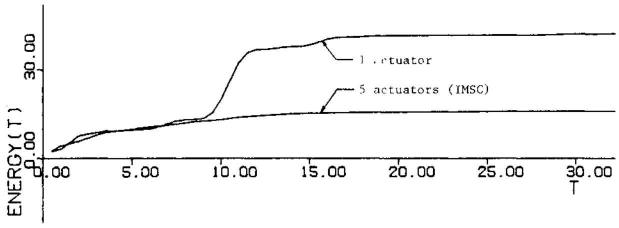

Fig. 2 Energy required for the controlled modes, pole allocation.

control force vector $f(t)$ [Eq. (30b)]. When coupled control is used, $R(t)$ is in terms of the actual control forces $F(t)$. The task is to select comparable weighting functions for coupled and independent controls.

For reasons stated earlier we choose $Q = \text{diag}(\omega_1^2\ \omega_1^2\ \omega_2^2\ \omega_2^2 \ldots \omega_n^2\ \omega_n^2)$ for both coupled and independent controls. The relation between the generalized and actual control forces is given by Eq. (12), from which it follows that the following quadratic terms are equivalent:

$$f^T R_2 f = F^T B^T R_2 B F \tag{53}$$

One can choose the weighting matrix for independent control R_2 and then calculate R_1. From Eq. (53), the weighting matrix for coupled control is

$$R_1 = B^T R_2 B \tag{54}$$

Note that, for the case of coupled control B is a rectangular matrix, so that one cannot compute R_2 from R_1. Hence, one cannot choose the weighting matrix for the actual control forces first and then determine the corresponding matrix for the modal control forces. In the case of independent control, R_2 is a diagonal matrix. Here, we choose it as

$$R_2 = \text{diag}\, R^* [1\ 1\ .\ .\ .\ 1] \tag{55}$$

where R^* is a weighting parameter.

Let us now consider the optimal control of 12 modes by using different numbers of actuators. The weighting parameter R^* is chosen as $R^* = 20$. We consider only the steady-state case of the optimal control problem. Table 2

Table 2 Real parts of the closed-loop poles for 12 controlled modes

m	4	6	8	10	12
$\text{Re}\lambda_1$	−0.088	−0.146	−0.151	−0.152	−0.152
$\text{Re}\lambda_2$	−0.088	−0.122	−0.154	−0.156	−0.157
$\text{Re}\lambda_3$	−0.105	−0.109	−0.154	−0.156	−0.158
$\text{Re}\lambda_4$	−0.098	−0.107	−0.152	−0.156	−0.158
$\text{Re}\lambda_5$	−0.050	−0.105	−0.148	−0.156	−0.158
$\text{Re}\lambda_6$	−0.114	−0.099	−0.123	−0.154	−0.158
$\text{Re}\lambda_7$	−0.106	−0.055	−0.106	−0.153	−0.158
$\text{Re}\lambda_8$	−0.092	−0.117	−0.100	−0.151	−0.158
$\text{Re}\lambda_9$	−0.081	−0.112	−0.057	−0.146	−0.158
$\text{Re}\lambda_{10}$	−0.044	−0.110	−0.117	−0.121	−0.158
$\text{Re}\lambda_{11}$	−0.095	−0.109	−0.113	−0.062	−0.158
$\text{Re}\lambda_{12}$	−0.090	−0.108	−0.111	−0.118	−0.158

Table 3 Computational effort required to solve the Riccati equation

	Number of actuators	Time, s
Coupled controls	4	176.5
	6	127.1
	8	104.5
	10	100.0
	12	87.7
IMSC	12	2.3

compares the real parts of the closed-loop poles obtained by solving the Riccati equation when 12 modes are controlled by using different numbers of actuators. Clearly, for comparable cost functions, the use of more actuators yields more stable closed-loop systems. Note that the closed-loop poles obtained by solving the Riccati equation for 12 actuators are identical to those obtained by IMSC. For the case in which two actuators were used, the closed-loop poles could not be computed because the Riccati equation solver[37] did not yield a convergent solution.

Table 3 compares the computational time required to solve the algebraic Riccati equation for different numbers of actuators with the time required for IMSC. Even though a closed-form solution exists for the determination of the control gains by IMSC, an iterative scheme was used here to compare the execution times. As can be seen from Table 3, there is a drastic difference in computational time between independent and coupled controls. In addition, the computational effort decreases as the number of actuators is increased.

The solution of the Riccati equation was obtained by using the subroutine MRIC[37] from a package of subroutines developed at MIT. The computations were carried out using the FORTRAN H extended compiler on an IBM 370-158 computer. Figure 3 compares the performance index $J(t)$ as time unfolds for different numbers actuators. Again, it is clear that the performance index decreases as the number of actuators is increased. The global minimum, as stated earlier, is obtained for controls comparable to those prescribed by the IMSC method.

Next, let us compare the response of the tip of the bar for the case when 4 and 12 actuators are used. As can be seen from Fig. 4, the vibration of the tip of the bar decays much faster when IMSC is used. This is due to two factors: 1) the magnitudes of the real parts of the closed-loop poles are larger when IMSC is used and 2) energy transfer from one mode to another due to coupling exists when four actuators are used.

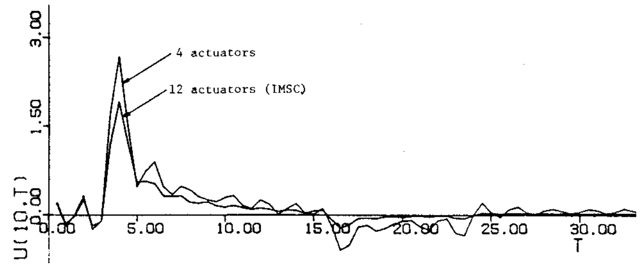

Fig. 3 Performance index, coupled and independent controls.

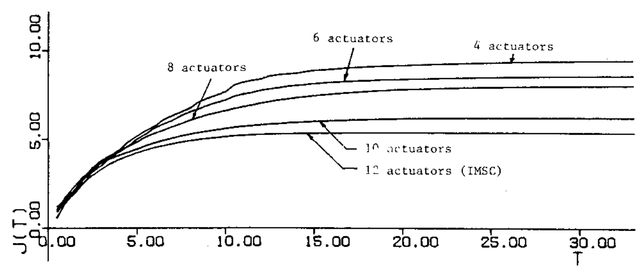

Fig. 4 Displacement at the end of the bar, optimal control.

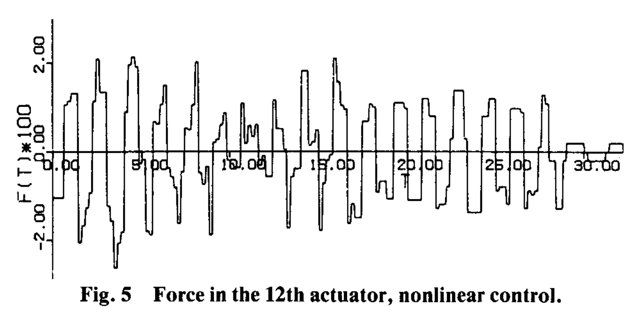

Fig. 5 Force in the 12th actuator, nonlinear control.

Nonlinear On-Off Control

For reasons mentioned earlier, nonlinear on-off controls cannot be designed for coupled control. Here we wish to demonstrate the design of on-off controls for IMSC. We control 12 modes and select the control gains k_r and the half-width of the deadband region d_r as follows:

$$k_r = 0.01, 0.02, 0.08, 0.05, 0.05, 0.05,$$
$$0.06, 0.07, 0.07, 0.08, 0.11, 0.11$$
$$d_r = 0.01, \qquad r = 1,2,\ldots,12$$

Figure 5 shows the force exerted by the 12th actuator on the beam. As can be seen, nonlinear on-off modal controls lead to quantized actual controls, so that their implementation is simpler than other controls. The fact that on-off controls can be used only in conjunction with IMSC is yet another advantage of this method.

Conclusions

Two approaches to the active control of flexible systems are compared. The first method, referred to as coupled control, does not place any requirements on the number of actuators, provided controllability is satisfied. The second method, referred to as independent modal-space control, requires that the number of actuators be equal to the number of controlled modes. The two methods are compared qualitatively from design and computational viewpoints and quantitatively considering input energy and performance index. It is concluded that the independent modal-space control method is superior to coupled control because independent control offers a larger choice of control techniques (including nonlinear control), is easier to design, exhibits virtually no instability characteristics, and minimizes the input energy. Also, it requires less computational effort and computer storage and can be implemented on microcomputers. In addition, independent modal-space control is relatively insensitive to the locations of the actuators.

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